

12 Geometric quantization

12.1 Remarks on quantization and representation theory

Definition 12.1 *Let M be a symplectic manifold. A prequantum line bundle with connection on M is a line bundle $\mathcal{L} \rightarrow M$ equipped with a connection ∇ for which the curvature F_∇ is equal to the symplectic form ω .*

Suppose a symplectic manifold M is equipped with a complex structure compatible with the symplectic structure (i.e. M is a Kähler manifold). Then if $(\mathcal{L}, \nabla)$ is a prequantum line bundle with connection, $\mathcal{L}$ naturally acquires a structure of holomorphic line bundle (we define the $\bar{\partial}$ operator on sections of $\mathcal{L}$ as the antiholomorphic part ∇'' of the prequantum connection, in other words a section s is holomorphic if and only if $\nabla'' s = 0$).

Definition 12.2 *Suppose M is a symplectic manifold equipped with a prequantum line bundle with connection $(\mathcal{L}, \nabla)$. The quantization of M is the virtual Hilbert space*

$$\mathcal{H}(M, \mathcal{L}) = \bigoplus_{i \text{ even}} H^i(M, \mathcal{L}) \ominus \bigoplus_{i \text{ odd}} H^i(M, \mathcal{L}).$$

Remark 12.3 *In many natural situations, only one of the vector spaces $H^i(M, \mathcal{L})$ is nonzero.*

Remark 12.4 *If M is compact, all the vector spaces $H^i(M, \mathcal{L})$ are finite-dimensional, and the dimension of the quantization is given by the Riemann-Roch theorem.*

Suppose M is equipped with a prequantum line bundle with connection, and suppose a group G acts in a Hamiltonian fashion on M , and that the group action lifts to the total space of $\mathcal{L}$ in a way that is compatible with the connection. (The choice of such a lift is in fact equivalent to the choice of a moment map for the group action: cf. Remark 8.44.) Then each of the vector spaces $H^i(M, \mathcal{L})$ is acted on by the group G , in other words the quantization of M is a (virtual) representation of G .

If M is acted on by a torus T , the multiplicities with which the weights for the action appear in the representation $\mathcal{H}$ of T are related to the moment polytope: all weights that have nonzero multiplicity lie within the moment polytope, and the asymptotics of the multiplicities of weights are in a natural sense given by the Duistermaat-Heckman polynomial f from Theorem 10.23. (For a precise statement, see Section 3.4 of [19].)

12.2 Integral closed 2-forms and line bundles

ω is integral iff for any cover $\{U_i\}$ of M there exists $\alpha_i \in \Omega^1(U_i)$ with $\omega|_{U_i} = d\alpha_i$ and $f_{ij} \in C^\infty(U_i \cap U_j)$ with $(\alpha_j - \alpha_i)|_{U_i \cap U_j} = df_{ij}$. Then on $U_i \cap U_j \cap U_k$,

$$f_{ij} + f_{jk} - f_{ik} = a_{ijk} \in \mathbb{R}$$

is a constant. $[\omega]$ is integral iff $a_{ijk} \in \mathbb{Z}$. We define transition functions

$$g_{jk} : U_j \cap U_k \rightarrow \mathbb{C}^*$$

by

$$g_{jk} = \exp i f_{jk}.$$

In order that these should satisfy

$$g_{ij}g_{jk} = g_{ik}$$

it is necessary that

$$f_{ij} + f_{jk} - f_{ik} \in 2\pi\mathbb{Z}.$$

On $M = S^2$, the usual symplectic form is $\omega = d\phi \wedge dz$. Take a different closed 2-form ω' on S^2 defined by $\omega' = d\phi \wedge df$ where $f : S^2 \rightarrow \mathbb{R}$ is a smooth function such that

(a) $f(z, \phi) = z$ for $z > 2\epsilon$ (b) $f(z, \phi) = -1$ for $z < -\epsilon$. Then $\int_{S^2} \omega' = 2\pi \int_{-1}^1 df = 4\pi$. In fact ω' is in the same class as ω in de Rham cohomology. so one is integral iff the other is. Take $U_0 = \{z < \epsilon\}$, $U_1 = \{z > -\epsilon\}$. On U_0 , $\omega'|_{U_0} = 0$ so $\omega' = d\alpha_0$ where $\alpha_0 = 0$. On U_1 , $\omega' = -d(fd\phi)$ so $\omega' = d\alpha_1$ where $\alpha_1 = -fd\phi$. On $U_0 \cap U_1$, $(\alpha_1 - \alpha_0)|_{U_0 \cap U_1} = df_{01} = -fd\phi = -d(f\phi) = d\phi$ since $f = -1$ on $U_0 \cap U_1$. Hence

$$f_{01}(z, \phi) = \phi$$

The effect on this calculation of replacing ω by $\lambda\omega$ (where λ is a constant) is that α_0, α_1 and f_{01} are multiplied by λ . So

$$g_{01}(z, \phi) = \exp i f_{01} = \exp i \lambda \phi.$$

This is single valued and defines a transition function iff $\lambda \in \mathbb{Z}$.

Definition 12.5 A symplectic form ω on a manifold M is integral if $[\omega] \in H^2(M, \mathbb{Z})$, or equivalently if for any oriented 2-dimensional submanifold S of M , $\int_S \omega \in \mathbb{Z}$.

Example 12.6 $\omega = dz$ on S^2 (where z is the height function): $\int_{S^2} \omega = 4\pi$ so

$$\frac{n\omega}{4\pi}$$

is integral for $n \in \mathbb{Z}$.

Lemma 12.7 ω is integral iff $[\omega] = c_1(L)$ for a line bundle L over M .

Remark 12.8 Chern classes $c_j(E)$ always take values in $H^{2j}(M, \mathbb{Z})$.

12.3 Prequantum line bundles with connection

Definition 12.9 A prequantum line bundle with connection over M is a complex line bundle L equipped with a connection ∇ whose curvature $iF_\nabla/(2\pi) = \omega$.

Definition 12.10 A connection ∇ on a line bundle L is a linear operator $\nabla : \Gamma(L) \rightarrow \Omega^1(M, L) = \Gamma(T^*M \otimes L)$ with the property that for a smooth function f and a section s on L ,

$$\nabla(fs) = (df)s + f\nabla s.$$

One may define $\nabla : \Omega^1(M, L) \rightarrow \Omega^2(M, L)$ where $\Omega^2(M, L) = \Gamma(\Lambda^2 T^*M \otimes L)$. Then

$$\nabla(\nabla s) = F_\nabla s$$

where $F_\nabla \in \Omega^2(M, \mathbb{C})$ is a 2-form (the curvature form).

12.4 Quantization and polarizations

In quantum mechanics, the standard phase space is $\mathbb{R}^{2n} = \{(q, p)\}$, the space of positions q and momenta p . One wants to pass from the phase space to the space of wave functions

$$\mathcal{H} = \{\Psi(q)\} = L^2(\mathbb{R}^n).$$

These are “functions of half the variables” (the Heisenberg uncertainty principle says that the more precisely one knows q , the less precisely one knows p , so the wave function Ψ , where $|\Psi|^2(q)$ is the probability of the particle being at position q , is a function only of q . One could equivalently write the wave function as $\hat{\Psi}(p)$, a function only of p . $\hat{\Psi}$ is the Fourier transform of Ψ .

Another way of defining “half the variables”: take $z_j = q_j + ip_j$ so $\mathbb{R}^{2n} = \mathbb{C}^n$ and define

$$\mathcal{H} = \{f : \mathbb{C}^n \rightarrow \mathbb{C} \mid f \text{ is holomorphic in the } z_j \text{ and } \int_{\mathbb{C}^n} e^{-|z|^2} f(z) dz_1 \dots dz_n < \infty\}.$$

In general, given a manifold M equipped with a prequantum line bundle with connection, in order to define a quantization one needs a polarization.

Definition 12.11 Real polarization (analogue of choice of $\{q\}$ or $\{p\}$ on $\mathbb{R}^{2n}$): choice of a foliation of M by Lagrangian submanifolds, in the case of $\mathbb{R}^{2n}$ these are $\{p = \text{const}\}$ or $\{q = \text{const}\}$.

Definition 12.12 Complex polarization: a choice of an almost complex structure J on M which is compatible with ω . We assume J is integrable i.e. comes from a structure of Kähler manifold on M .

12.5 Holomorphic line bundles

Holomorphic line bundle over a complex manifold:

Definition 12.13 A complex line bundle is specified by an open cover $\{U_\alpha\}$ on M and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^*$ and $L = \cup_\alpha U_\alpha \times \mathbb{C} / \sim$ where $(x, z_\alpha) \cong (x, z_\beta)$ if $z_\alpha = g_{\alpha\beta}(x)z_\beta$.

Definition 12.14 The line bundle is holomorphic if the transition functions $g_{\alpha\beta}$ are holomorphic.

Definition 12.15 A section s of L is a collection of maps $s_\alpha : U_\alpha \rightarrow \mathbb{C}$ satisfying $s_\alpha(z) = g_{\alpha\beta}(z)s_\beta(z)$ for $z \in U_\alpha \cap U_\beta$. (This makes sense since $\frac{\partial}{\partial \bar{z}_j} g_{\alpha\beta} = 0$ so on $U_\alpha \cap U_\beta$ $\frac{\partial}{\partial \bar{z}_j} s_\alpha = 0$ iff $\frac{\partial}{\partial \bar{z}_j} s_\beta = 0$).

Definition 12.16 Complex (co) tangent space:

$$T_{\mathbb{C}}M = TM \otimes \mathbb{C}$$

$$T_{\mathbb{C}}^*M = T^*M \otimes \mathbb{C}$$

In local complex coordinates z_j , a basis for $T_{\mathbb{C}}^*M$ is $\{dz_j, d\bar{z}_j, j = 1, \dots, n$.

Definition 12.17 Holomorphic and antiholomorphic cotangent spaces

$$T_{\mathbb{C}}^*M = (T^*)^{(1,0)}M \oplus (T^*)''M$$

where in local complex coordinates $(T^*)''M$ is spanned by $\{d\bar{z}_j\}$ and $(T^*)'M$ is spanned by $\{dz_j\}$.

Definition 12.18 $\bar{\partial}$ -operator on functions on M

Choose local complex coordinates $z_1, \dots, z_n$ on the U_α and define

$$\bar{\partial} : C^\infty(U_\alpha) \rightarrow \Omega^{0,1}(U_\alpha)$$

$$\bar{\partial}f = \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

Definition 12.19 ($\bar{\partial}$ operator on sections of L on M) Given a section $s : M \rightarrow L$, $s = \{s_\alpha\}$, define $\bar{\partial}s \in \Gamma(T^*)''M \otimes L$ by

$$\bar{\partial}s = \bar{\partial}s_\alpha$$

on U_α .

This is well defined since $\bar{\partial}g_{\alpha\beta} = 0$.

Proposition 12.20 *Specifying a structure of holomorphic line bundle on a complex line bundle L is equivalent to specifying an operator $\bar{\partial} : \Gamma(L) \rightarrow \Omega^{0,1}(M, L)$ satisfying $\bar{\partial} \circ \bar{\partial} = 0$.*

Proof: We have seen that a holomorphic line bundle determines a $\bar{\partial}$ operator. Conversely, given a complex line bundle L with $\bar{\partial}$, we can choose an open cover $\{U_\alpha\}$ with locally defined solutions $s_\alpha \in \Gamma(L|_{U_\alpha})$ to $\bar{\partial}s_\alpha = 0$, and $s_\alpha(x) \neq 0 \forall x \in U_\alpha$. Define transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^*$ by

$$g_{\alpha\beta} = s_\alpha s_\beta^{-1}.$$

It follows that $\bar{\partial}g_{\alpha\beta} = 0$ so $g_{\alpha\beta}$ gives L the structure of a holomorphic bundle. $\square$

Proposition 12.21 *Let (L, ∇) be a prequantum line bundle over M . Suppose M is equipped with a complex structure J compatible with ω . (in other words, on M there are locally defined complex coordinates $\{z_j\}$). Then $\nabla : \Gamma(L) \rightarrow \Gamma((T^*)M \otimes L)$ decomposes as*

$$\nabla = \nabla'' \oplus \nabla'$$

where

$$\nabla'' : \Gamma(L) \rightarrow \Gamma((T^*)''M \otimes L)$$

and

$$\nabla' : \Gamma(L) \rightarrow \Gamma((T^*)'M \otimes L).$$

Note that ∇'', ∇' depend on the almost complex structure J on M .

Proposition 12.22 *We may define a structure of holomorphic line bundle on L by defining ∇'' as a $\bar{\partial}$ operator: a section s of L is defined to be holomorphic if*

$$\nabla'' s = 0.$$

Definition 12.23 *The quantization of the symplectic manifold (M, ω) equipped with the prequantum line bundle L with connection ∇ and the complex structure J is*

$$\mathcal{H} = H^0(M, L),$$

in other words the global holomorphic sections of L .

Remark 12.24 *If M is compact, $\mathcal{H}$ is a finite-dimensional complex vector space.*

12.6 Quantization of $\mathbb{C}P^1 \cong S^2$

$$\begin{aligned}\mathbb{C}P^1 &= \{(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}\} / \sim \\ &= \{[z_0 : z_1]\}\end{aligned}$$

The hyperplane line bundle over $\mathbb{C}P^1$ is $L_{[z_0 : z_1]} = \{f : \{\lambda(z_0, z_1) \rightarrow \mathbb{C}\}$

$$f(z) = f_0 z_0 + f_1 z_1$$

Its dual is the tautological line bundle

$$L_{[z_0 : z_1]}^* = \{(\lambda z_0, \lambda z_1) : \lambda \in \mathbb{C}\}$$

This is the line through the point (z_0, z_1) . The k -th power of the tautological bundle is

$$L_{[z_0, z_1]}^k = \{f : \{(\lambda z_0, \lambda z_1) \rightarrow \mathbb{C} : f(\lambda z_0, \lambda z_1) = \lambda^k f(z_0, z_1)\}$$

in other words f is a polynomial of degree k on the line through $(z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$. Its zero-th power is the trivial bundle $L^0 = \mathbb{C}P^1 \times \mathbb{C}$.

Global holomorphic sections:

$H^0(L)$ is spanned by the restrictions to $\mathbb{C}^2 \setminus \{0\}$ of the linear functions on $\mathbb{C}^2$. This is a space of dimension 2. $H^0(L^k)$ is spanned by the restrictions to $\mathbb{C}^2 \setminus \{0\}$ of the polynomials of degree k on $\mathbb{C}^2$:

$$f(z_0, z_1) = \sum_{j=0}^k a_j z_0^j z_1^{k-j}.$$

This is a space of dimension $k + 1$.

12.7 Link to representation theory

Suppose a (compact) group G acts on M (from the left), preserving the complex structure J as well as the symplectic structure (in other words, for each $g \in G$, $L_g : M \rightarrow M$ is a holomorphic diffeomorphism).

Suppose the G action lifts to an action on the total space L of a prequantum line bundle which preserves the connection ∇ , and that this action is linear in the fibres: in other words

$$L_g : \pi^{-1}(m) \rightarrow \pi^{-1}(gm)$$

is a linear map.

Proposition 12.25 *In this situation, the G action defines an action of G on $\mathcal{H}$ (from the right).*

Define $(s \cdot g)(m) = s(g(M))$, in other words $s \cdot g = s \circ L_g$. Thus since L_g is a holomorphic diffeomorphism, the composition $s \circ L_g$ is a holomorphic section.

Proposition 12.26 *The action of G on the space of holomorphic sections is linear. Thus $\mathcal{H}$ is a linear representation of G*

Proof: $(s_1 + s_2) \cdot g = s_1 \cdot g + s_2 \cdot g$. □

Proposition 12.27 *Let M be a symplectic manifold acted on by T , and suppose ω is an integral symplectic form. Then the weights $\beta \in \mathfrak{g}^*$ of the representation of T on $\mathcal{H}$ lie in the moment polytope $\Phi_T(M) \subset \mathfrak{t}^*$. These will in general appear with some multiplicities m_β , in other words $\mathcal{H} = \bigoplus_{\beta \in \Lambda^W} m_\beta \mathbb{C}_\beta$, $m_\beta \in \mathbb{Z}^+$. (This is given by the Kostant multiplicity formula, and its generalizations due to Guillemin.)*

Remark 12.28

1. For toric manifolds, a weight appears with multiplicity 1 iff it is in $\Phi(M)$ (and 0 otherwise).
2. The multiplicity function $m : \Lambda^W \rightarrow \mathbb{Z}^{\geq 0}$ is related to the pushforward $\frac{\Phi_* \omega^n}{n!}$. The pushforward is obtained from the asymptotics of the multiplicity function under replacing ω by $k\omega$, $k \in \mathbb{Z}^+$ (this operation dilates the moment polytope by k).

12.8 Holomorphic bundles over G/T : the Borel-Weil theorem

Theorem 12.29 (Kostant) *Suppose $\lambda \in \mathfrak{t}^*$. The symplectic form ω on the coadjoint orbit $\mathcal{O}_\lambda$ is integral iff $\lambda \in \Lambda^W \subset \mathfrak{t}^*$.*

Let $\lambda \in \Lambda^W$, $\text{Stab}(\lambda) = T$. We may define a complex line bundle L_λ over $G/T \cong \mathcal{O}_\lambda$ as follows.

$$\rho_\lambda = \exp \lambda \in \text{Hom}(T, U(1))$$

so define

$$L_\lambda = G \times_{T, \rho_\lambda} \mathbb{C}$$

$= (G \times \mathbb{C}) / \sim$ where

$$(g, z) \sim (gt^{-1}, \rho_\lambda(t)z).$$

Sections of L_λ are given by equivariant maps $G \rightarrow \mathbb{C}$

$$= \{f : G \rightarrow \mathbb{C} \mid f(gt^{-1}) = \rho_\lambda(t)f(g)\}$$

The action of G on the space of sections is

$$g \cdot f(hT) = f(ghT).$$

Proposition 12.30 $G/T = G^{\mathbb{C}}/B$ where $G^{\mathbb{C}}$ is the complexification of G and B (Borel subgroup) is a complex Lie group defined by

$$\mathrm{Lie}(B) = (\mathrm{Lie}(T) \otimes \mathbb{C}) \oplus \bigoplus_{\gamma > 0} \mathbb{C}\gamma.$$

Recall that $\mathrm{Lie}(G) \otimes \mathbb{C}$ decomposes under the adjoint action of T as

$$(\mathrm{Lie}(T) \otimes \mathbb{C}) \oplus \bigoplus_{\gamma > 0} \mathbb{C}\gamma \oplus \bigoplus_{\gamma > 0} \mathbb{C}_{-\gamma}.$$

Examples of complexifications of Lie groups:

$$SU(n)^{\mathbb{C}} = SL(n, \mathbb{C})$$

$$U(1)^{\mathbb{C}} = \mathbb{C}^*$$

$$U(n)^{\mathbb{C}} = GL(n, \mathbb{C})$$

Examples of Borel subgroups:

$$G = U(n)$$

$$G^{\mathbb{C}} = GL(n, \mathbb{C})$$

B is the set of upper triangular matrices in $GL(n, \mathbb{C})$ (in other words $z_{ij} = 0$ if $i > j$).

The groups $G^{\mathbb{C}}$ and B have obvious complex structures: so, therefore, does $G^{\mathbb{C}}/B$. This holomorphic structure is compatible with ω_{λ} (it comes from the complex structure J on $\mathrm{Lie}(G) \otimes \mathbb{C}$).

$$\omega_{\lambda}([\lambda, X], [\lambda, Y]) = \langle \lambda, [X, Y] \rangle.$$

gives $\omega_{\lambda}(JZ_1, JZ_2) = \omega_{\lambda}(Z_1, Z_2)$. Here, the almost complex structure J is defined on $T_{\lambda}(G/T)$ and is defined at $T_{g \cdot \lambda}(G/T)$ by identifying this with $T_{\lambda}(G/T) \cong \bigoplus_{\gamma > 0} \mathbb{C}\gamma$. It is integrable.

Thus L_{λ} acquires the structure of a holomorphic line bundle.

Lemma 12.31 *There is a homomorphism $p : B \rightarrow T_{\mathbb{C}}$.*

Proof: B has a normal subgroup $N_{\mathbb{C}}$ for which $T_{\mathbb{C}} = B/N_{\mathbb{C}}$. □

Example 12.32 $GL(n, \mathbb{C})$

$T_{\mathbb{C}}$ is the invertible diagonal matrices

B is the upper triangular matrices

p is projection on the diagonal

Hence $\rho_\lambda = \exp(\lambda) : T \rightarrow U(1)$ extends to $\rho_\lambda : T_{\mathbb{C}} \rightarrow \mathbb{C}^*$ and to $\bar{\rho}_\lambda : B \rightarrow \mathbb{C}^*$ via $\bar{\rho}_\lambda = \rho_\lambda \circ p$. Thus we can define

$$\begin{aligned} L_\lambda &= G_{\mathbb{C}} \times_{B, \rho} \mathbb{C} \\ &= \{(g, z)\} / \sim \end{aligned}$$

where $(g, z) \sim (gb^{-1}, \rho_\lambda(b)z)$ for all $b \in B$.

The space of holomorphic sections of L_λ is

$$H^0(\mathcal{O}_\lambda, L_\lambda) = \{f : G^{\mathbb{C}} \rightarrow \mathbb{C} : f \text{ holo.}, f(gb^{-1}) = \rho_\lambda(b)f(g)\}$$

for all $g \in G^{\mathbb{C}}$ and $b \in B$.

Theorem 12.33 (Borel-Weil-Bott) : *If $\lambda \in \Lambda^W$ is in the positive Weyl chamber, then $H^0(\mathcal{O}_\lambda, L_\lambda)$ is the irreducible representation of G with highest weight λ .*

Representations of $SU(2)$:

The representations of $SU(2)$ arise by quantizing S^2 .

$$\begin{aligned} H^0(M, L) &= \{a_0 z_0 + a_1 z_1\} \\ H^0(M, L^k) &= \left\{ \sum_j a_j z_0^j z_1^{k-j} \right\} \\ \tau &:= \text{diag}(t, t^{-1}) \in SU(2) \end{aligned}$$

acts on $\mathbb{C}^2$ by sending

$$\tau : \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \mapsto \begin{pmatrix} tz_0 \\ t^{-1}z_1 \end{pmatrix}$$

So $z_0^{k-j} z_1^j \mapsto t^{k-2j} z_0^{k-j} z_1^j$

There are $k+1$ weights in total, each appearing with multiplicity 1.

Roots:

1. Decompose $\text{Lie}(G) \otimes \mathbb{C}$ under the adjoint action of the maximal torus T . The roots are the weights of this action of T . They appear in pairs (if β is a root, so is $-\beta$).
2. Choose a polarization to enable us to designate some roots β positive, while $-\beta$ is designated as negative.
3. *Simple roots* are a collection of roots which form a basis of $\text{Lie}(T)$.

Example 12.34

$$\begin{aligned} &SU(n) \\ \text{Lie}(T) &= \{\text{diag}(X_1, \dots, X_n) \mid \sum_j X_j = 0\} \end{aligned}$$

The roots are $\gamma_{ij}(X) = X_i - X_j$, and the positive roots are γ_{ij} with $i < j$. The simple roots are $\gamma_{12}, \dots, \gamma_{(n-1)n}$. The positive Weyl chamber consists of the subset of $\mathfrak{t}$ for which the inner product with all simple roots is > 0 .